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A TWO-PRODUCT PERISHABLE/NONPERISHABLE INVENTORY PROBLEM*

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Abstract. We consider a situation in which two types of inventories are available to satisfy demands, one having finite lifetime and one an infinite lifetime. It is assumed that demands form a sequence of independent random variables which first deplete from the perishable inventory and then the nonperishable. We show that there are exactly three ordering regions in each period which correspond to the three alternatives: ordering in both periods, ordering only perishable inventory or not ordering. The region boundaries and the optimal policies are characterized for both the single period and dynamic problems. A unique property of the model is that even with inclusion of salvage values at the end of the horizon, the region boundaries remain nonstationary.

1. Introduction. A problem which is beginning to receive attention in the literature is that of describing optimal ordering policies for a commodity with a fixed lifetime. The structure of the optimal policy becomes significantly more complex when the inventory is perishable in nature. Analysis of the single product case for a product with a lifetime of two periods was initiated by Van Zyl [8] and later by Nahmias and Pierskalla [3]. Fries [2], Nahmias and Pierskalla [4] and Nahmias [5] have considered the extension of these models to a product with an arbitrary but fixed lifetime. The purpose of this paper is to generalize the model considered by Nahmias [5] and consider a system consisting of one fixed life perishable product and one nonperishable, where the nonperishable may be substituted for the perishable.

Our intent is to develop a more realistic model of inventory management for a central blood bank. As units of blood are drawn from donors they may be refrigerated, in which case the shelf life is twenty-one days, or frozen, in which case shelf life is 365 days, which for all practical purposes is nonperishable. When demands exceed the available supply of fresh blood, frozen blood may then be thawed. Another possible application of this model is in the area of food management: a military supply depot stocking both fresh and powdered milk would face a similar problem.

2. The one-period model. We make the following assumptions:

(1) All orders are placed at the start of a period and received instantly.

(2) All stock arrives new.

(3) Demands in successive periods are independent identically distributed random variables with distribution F and density f. In addition we assume that $f(t) > 0$ when $t > 0$.

(4) Inventory of product 1 (the perishable product) is depleted according to a FIFO policy, that is, first in first out.

(5) All costs are linear. They include

(a) Ordering in both products (at unit costs c_1 and c_2) charged at the start of the period;

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(b) Holding in both products (at unit costs h_1 and h_2) charged on what is on hand at the end of the period;

(c) Shortage (at unit cost r) charged on the unsatisfied demand at the end of the period;

(d) Outdating (at unit cost θ) charged on what deteriorates at the end of the period.

(6) If product 1 has not been depleted by demand before reaching age m-periods, it must be discarded at the unit cost given in 5(d).

(7) There is a single demand source. Demands first deplete from product 1 (the perishable product) and then product 2. Excess demand is backlogged in the second product.

A number of our assumptions may be relaxed or altered; generalizations will be discussed in the final section. It should be noted that assumption (4) is quite mild. The optimality of FIFO for depletion of perishable inventory has been established under far more general circumstances (Nahmias [6] and Pierskalla and Roach [7]).

Our approach will be to charge the outdating cost against the expected outdating of the present order which will not occur for m periods. The motivation behind this method is discussed in [5]. If $\mathbf{x} = (x_{m-1}, \cdots, x_1)$ is the vector of perishable inventory on hand, x_i = number of units on hand which will outdate in exactly i periods, and y is the amount of new perishable inventory ordered, then it has been shown in [5] that the expression $\int_0^y G_m(u;\mathbf{x})\,du$ represents the expected outdating of y, m periods into the future, where

$$G_n(t;\mathbf{x}(n-1)) = \int_0^t G_{n-1}(x_{n-1}+v;\mathbf{x}(n-2))f(t-v)\,dv$$

for $1 \leqq n \leqq m$; $\mathbf{x}(n) = (x_n, \cdots, x_1)$ and

$$G_0(t) = \begin{cases} 1 & \text{if } t \geqq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

For each $n \geqq 1$, $G_n(t;\mathbf{x}(n-1))$ is a C.D.F. in its first argument, and may possess a discontinuity at $t = 0$.

Letting

x^2 = amount of product 2 on hand,

z = amount of product 2 on hand after ordering,

the total expected cost of ordering y of product 1 and $z - x^2$ of product 2 is

$$c_1 y + h_1 \int_0^{x+y} (x+y-t)f(t)\,dt + \theta \int_0^y G_m(u;\mathbf{x})\,du + c_2(z - x^2) + h_2 z F(x+y)$$
$$+ h_2 \int_{x+y}^{x+y+z} (x+y+z-t)f(t)\,dt + r \int_{x+y+z}^{\infty} (t-x-y-z)f(t)\,dt,$$

where $x = \sum_{i=1}^{m-1} x_i$ for convenience. Collecting all terms independent of x^2, we will write this as $L(\mathbf{x}, y, z) - c_2 x^2$. A point which should be noted here is that the decision variables for each product have different interpretations: y represents the actual quantity of product 1 ordered, while z is the inventory level of product 2 after ordering. Our interest in the single period model is secondary to that of the multidecision dynamic problem. The optimal ordering policy over the finite horizon will satisfy the functional equations

$$C_n(\mathbf{x}, x^2) = \inf_{\substack{y \geqq 0 \\ z \geqq x^2}} \left\{ L(\mathbf{x}, y, z) - c_2 x^2 + \alpha \int_0^\infty C_{n-1}(\mathbf{s}_1(\mathbf{x}, y, t), s_2(x + y, z, t)) f(t)\, dt \right\}$$

for $1 \leqq n \leqq N$ (N is a fixed positive integer). The transfer functions are given by

$$\mathbf{s}_1(\mathbf{x}, y, t) = (s_{1,m-1}(\mathbf{x}, y, t), \cdots, s_{1,1}(\mathbf{x}, y, t)),$$

where

$$s_{1,i}(\mathbf{x}, y, t) = \left[x_{i+1} - \left(t - \sum_{j=1}^{i} x_j \right)^+ \right]^+, \qquad 1 \leqq i \leqq m - 1,$$

(we interpret $x_m = y$) and $s_2(x + y, z, t) = z - (t - (x + y))^+$, where $g^+ = \max(g, 0)$. The discount factor is $\alpha \in (0, 1)$.

Again collecting all terms independent of x^2, we write

$$C_n(\mathbf{x}, x^2) = \inf_{\substack{y \geqq 0 \\ z \geqq x^2}} \{B_n(\mathbf{x}, y, z) - c_2 x^2\},$$

which implies

$$B_n(\mathbf{x}, y, z) = L(\mathbf{x}, y, z) + \alpha \int_0^\infty C_{n-1}(\mathbf{s}_1(\mathbf{x}, y, t), s_2(x + y, z, t)) f(t)\, dt.$$

The functions $C_n(\mathbf{x}, x^2)$ have the usual interpretation as the minimum expected discounted cost for an n period problem when $(\mathbf{x}, x^2)$ is on hand. As is customary with dynamic programming models, the periods are numbered backwards. The goal of our analysis will be to answer the two questions: when should an order be placed and how much of each product should be ordered?

3. The ordering regions. It becomes convenient to introduce the assumption that inventory remaining at the end of the horizon can be salvaged: inventory of product 1 remaining may be salvaged at a return $c_1 x$ and of product 2 at a return $c_2 x^2$. Backlogged demand in product 2 may be made up by an emergency order at a cost of $-c_2 x^2$. That is,

$$C_0(\mathbf{x}, x^2) = -c_1 x - c_2 x^2.$$

The assumption is identical to that made by Veinott [9] in the analysis of nonperishable multiproduct problems.

In addition we make the following four assumptions regarding the cost parameters:

(i) $0 \leqq h_2 \leqq h_1$,

(ii) $0 < c_1 < c_2$,

(iii) $r > (1 - \alpha)c_2$,

(iv) $0 \leqq (1 - \alpha)(c_2 - c_1) + (h_2 - h_1) < \theta$.

Since perishable inventory must often be stored under special conditions, assumption (i) is not unreasonable. Assumption (ii) is necessary to insure that it is economical to stock product 1. Assumption (iii) is a usual one made for nonperishable inventory [1]. The expression of assumption (iv) is precisely the cost of procuring one unit of product 2, holding it for one period and salvaging it the following period, minus the cost of procuring one unit of product 1, holding it for one period and salvaging it the following period. If this term were negative, then it would never be optimal to order in product 1, while if it exceeded the unit cost of outdating, it would never be optimal to order to a positive level in product 2.

We shall need to refer to the following constants in the analysis of the ordering regions in the first period:

$$u^* = F^{-1}\left[\frac{r - c_2(1 - \alpha)}{r + h_2}\right],$$

$$w^* = F^{-1}\left[\frac{r - c_1(1 - \alpha) + (h_2 - h_1)}{r + h_2}\right].$$

The constant u^* corresponds to the optimal critical number for a single product nonperishable problem [9]. Both constants, u^* and w^*, will be needed to describe the ordering regions.

When F is strictly increasing, assumptions (iii) and (iv) guarantee that u^* and w^* both exist and are strictly positive. If the additional condition $\alpha c_2 - h_2 < c_1$ is satisfied (although we will not require it to be), then we may also define the constant

$$v^* = F^{-1}\left[\frac{r - c_1 + \alpha c_2}{r + h_2}\right].$$

With the inclusion of the salvage value assumption, it follows from the definition of the transfer function that

$$\begin{aligned} B_1(\mathbf{x}, y, z) = {} & L(\mathbf{x}, y, z) + \alpha F(x_1)\left[-c_1\left(y + \sum_{i=2}^{m-1} x_i\right) - c_2 z\right] \\ & - \alpha c_1 \int_{x_1}^{x+y} (x + y - t) f(t)\, dt - \alpha c_2 [F(x + y) - F(x_1)] z \\ & - \alpha c_2 \int_{x+y}^{x} (x + y + z - t) f(t)\, dt. \end{aligned}$$

We adopt the following notational convention: If $h: R^n \to R^1$ and $h \in C^{(2)}$, then $h^{(i)}$ is the first partial derivative of h with respect to its ith argument and $h^{(i,j)}$

is the second cross partial derivative with respect to the ith and jth arguments respectively.

We have the following.

THEOREM 3.1. *$B_1(\mathbf{x}, y, z)$ is convex in (y, z) for all nonnegative $\mathbf{x}$. The functions $y_1(\mathbf{x})$, $z_1(\mathbf{x})$ solving $B_1(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) = \min_{y,z}\{B_1(\mathbf{x}, y, z)\}$ satisfy $B_1^{(m)}(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) = B_1^{(m+1)}(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) = 0$ and are unique for all $\mathbf{x}$.*

Proof.

$$B_1^{(m,m)}(\mathbf{x}, y, z) = \alpha(c_2 - c_1)f(x + y) + \theta G_m^{(1)}(y; \mathbf{x}) + (h_1 - h_2)f(x + y) + (r + h_2)f(x + y + z) \geqq 0$$

(note: $G_m^{(1)}(y, \mathbf{x})$ is a p.d.f.) and $B_1^{(m+1,m+1)}(\mathbf{x}, y, z) = B_1^{(m,m+1)}(\mathbf{x}, y, z) = (r + h_2) \cdot f(x + y + z) \geqq 0$ and convexity follows since $B_1^{(m,m)}(\mathbf{x}, y, z) \geqq B_1^{(m,m+1)}(\mathbf{x}, y, z)$. The minimizing point $(y_1(\mathbf{x}), (z_1(\mathbf{x}))$ satisfies the two equations

$$B_1^{(m)}(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) = 0,$$
$$B_1^{(m+1)}(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) = 0,$$

which will be consistent by assumption (iii). Combining the results of these equations, we obtain

$$c_1(1 - \alpha F(x + y_1(\mathbf{x}))) - c_2(1 - \alpha F(x + y_1(\mathbf{x}))) + \theta G_m(y_1(\mathbf{x}); \mathbf{x}) + (h_1 - h_2)F(x + y_1(\mathbf{x})) = 0.$$

Assumption (iii) now guarantees that $0 < y_1(\mathbf{x}) < +\infty$ for all $0 \leqq x < +\infty$ so that $x + y_1(\mathbf{x}) > 0$ and $B_1(\mathbf{x}, y, z)$ is strictly convex at the point $(y_1(\mathbf{x}), z_1(\mathbf{x}))$, from which uniqueness follows. □

Since $y_1(\mathbf{x}) > 0$ for all $\mathbf{x}$, a necessary and sufficient condition that it be possible to order to the global minimum of $B_1(\mathbf{x}, y, z)$ is $x^2 < z_1(\mathbf{x})$. Because of the convexity of $B_1(\mathbf{x}, y, z)$ in (y, z) (and hence the convexity in z), when $x^2 \geqq z_1(\mathbf{x})$ it will be optimal not to order product 2. However, since y represents an order quantity (rather than an order up to the point), it may still be optimal to order product 1 when $x^2 \geqq z_1(\mathbf{x})$. In this case we define the function $p_1(\mathbf{x}, x^2)$ to satisfy

$$B_1(\mathbf{x}, p_1(\mathbf{x}, x^2), x^2) \equiv \inf_y (B_1(\mathbf{x}, y, x^2)).$$

If $p_1(\mathbf{x}, x^2) > 0$, then it is optimal to order this amount of product 1. Hence the following characterization holds.

THEOREM 3.2. *A necessary and sufficient condition that it is optimal to order a positive amount of product 1 is $B_1^{(m)}(\mathbf{x}, 0, x^2) < 0$.*

Proof. If $x^2 < z_1(\mathbf{x})$, then from the proof of Theorem 3.1, $y_1(\mathbf{x}) > 0$ so that $0 = B_1^{(m)}(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) > B_1^{(m)}(\mathbf{x}, 0, x^2)$.

If $x^2 \geqq z_1(\mathbf{x})$, then a necessary and sufficient condition that it be optimal to order a positive amount in product 1 is that $p_1(\mathbf{x}, x^2) > 0$, which is true if and only if $B_1^{(m)}(\mathbf{x}, 0, x^2) < 0$. □

If one reasons analogously to Theorem 3.2, then it is tempting to assume that $B_1^{(m+1)}(\mathbf{x}, 0, x^2) < 0$ implies it is optimal to order in product 2. However, this is not the case. To see why, let $t(\mathbf{x}, y)$ satisfy $B_1^{(m+1)}(\mathbf{x}, y, t(\mathbf{x}, y)) = 0$. Differentiating implicitly with respect to y, we obtain

$$t^{(m)}(\mathbf{x}, y) = \frac{-B_1^{(m+1,m)}(\mathbf{x}, y, t(\mathbf{x}, y))^{\bullet}}{B_1^{(m+1,m+1)}(\mathbf{x}, y, t(\mathbf{x}, y))} < 0.$$

Since $t(\mathbf{x}, y_1(\mathbf{x})) = z_1(\mathbf{x})$ and $y_1(\mathbf{x}) > 0$, it follows that $t(\mathbf{x}, 0) > z_1(\mathbf{x})$. If x^2 satisfies $z_1(\mathbf{x}) \leqq x^2 < t(\mathbf{x}, 0)$, then $B_1^{(m+1)}(\mathbf{x}, 0, x^2) < 0$ and it is optimal not to order in product 2.

Hence there are exactly three distinct ordering regions:

Region I—Optimal to order in both products: $x^2 < z_1(\mathbf{x})$,

Region II—Optimal to order in product 1 only: $x^2 \geqq z_1(\mathbf{x})$ and

$$B_1^{(m)}(\mathbf{x}, 0, x^2) < 0 \quad (p_1(\mathbf{x}, x^2) > 0),$$

Region III—Optimal not to order: $B_1^{(m)}(\mathbf{x}, 0, x^2) \geqq 0$, $(p_1(\mathbf{x}, x^2) \leqq 0)$.

Note from the proof of Theorem 3.2 that if it is optimal to order in product 2, then it is optimal to order in product 1. The boundary between regions I and II is quite complex, as it depends on the entire vector $(\mathbf{x}, x^2)$. However, the boundary between regions II and III depends on the vector $\mathbf{x}$ only through the sum of its components, x.

Define $g(x) = (1/(r + h_2)) \cdot \{r + \alpha c_2 - c_1 - F(x)[(h_1 - h_2) + \alpha(c_2 - c_1)]\}$. Then we have the following.

THEOREM 3.3. *A necessary and sufficient condition that* $(\mathbf{x}, x^2)$ *is in Region* I *or* II *is that*

$$F(x + x^2) < g(x).$$

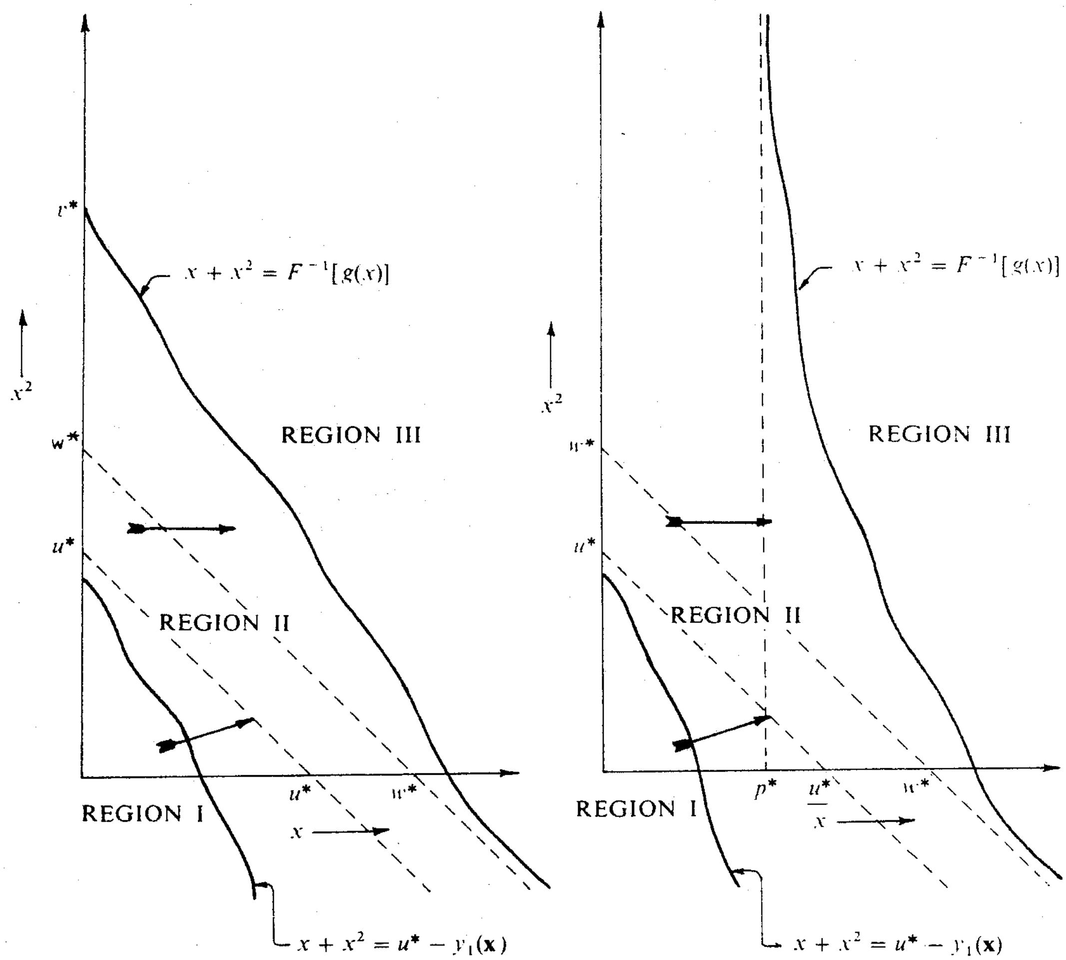
Proof. From Theorem 3.2 and the definitions above, it must be true that $B_1^{(m)}(\mathbf{x}, 0, x^2) < 0$.

$$\begin{aligned} B_1^{(m)}(\mathbf{x}, 0, x^2) &= c_1(1 - \alpha F(x)) + (h_1 - h_2)F(x) - r \\ &\quad + (r + h_2)F(x + x^2) - \alpha c_2(1 - F(x)) \\ &= (r + h_2)(F(x + x^2) - g(x)). \end{aligned}$$

The result then follows. □

The function $g(x)$ is strictly decreasing in x with $g(+\infty) = F(w^*)$. If the condition $\alpha c_2 - h_2 < c_1$ is satisfied, then $F(w^*) \leqq g(x) < 1$ for all $x \geqq 0$, and $v^* = F^{-1}(g(0))$ will exist. However, if $\alpha c_2 - h_2 \geqq c_1$, then $g(0) \geqq 1$. In this case, there exists a unique number $p^* \geqq 0$ which solves $g(p^*) = 1$. The boundary between Regions II and III may be pictured in the (x, x^2)-plane independent of m. Figure 1 pictures this boundary for each of the two cases above. The boundary between Regions I and II may be pictured in the (x, x^2)-plane only if $m = 2$. The arrows indicate the inventory position after ordering. Notice that $(\mathbf{x}, x^2) \in$ Region I guarantees that $x + y_1(\mathbf{x}) + z_1(\mathbf{x}) = u^*$ (which follows from $B_1^{(m+1)}(\mathbf{x}, y_1(\mathbf{x}), z_1(\mathbf{x})) = 0$), while $(x, x^2) \in$ Region II will yield $F(x + x^2 + p_1(\mathbf{x}, x^2)) < g(x + p_1(\mathbf{x}, x^2))$ as will be demonstrated.

In the next section we extend these results to the multiperiod dynamic problem. A thorough investigation of the structure of the optimal policies is undertaken.



[a] $\alpha c_1 - h_2 < c_1$ [b] $\alpha c_2 - h_2 \geqq c_1$

FIG. 1. *The optimal ordering regions for the one-period problem when $m = 2$*

Even with the inclusion of the salvage value assumption, neither the ordering regions nor the ordering policies is stationary in time.

4. The dynamic problem. Recall that the functional equations defining an optimal policy are

$$C_n(\mathbf{x}, x^2) = \inf_{\substack{y \geqq 0 \\ z \geqq x^2}} \{B_n(\mathbf{x}, y, z) - c_2 x^2\},$$

where

$$B_n(\mathbf{x}, y, z) = L(\mathbf{x}, y, z) + \alpha \int_0^\infty C_{n-1}(\mathbf{s}_1(\mathbf{x}, y, t) s_2(x + y, z, t)) f(t)\, dt.$$

We will denote by $y_n(\mathbf{x})$ the optimal quantity of product 1 to be ordered when $\mathbf{x}$ is on hand and n periods remain and by $z_n(\mathbf{x})$ the optimal quantity of product 2 to be ordered when $(\mathbf{x}, x^2) \in$ Region I. Extending the notation introduced in the previous section in a similar fashion, we denote by $p_n(\mathbf{x}, x^2)$ the optimal quantity of product 1 to be ordered when $(\mathbf{x}, x^2) \in$ Region II and n periods remain.

As in the single-period case, $z_n(\mathbf{x})$ will determine the boundary between Regions I and II. However, to characterize the boundary between Regions II and

III when n periods remain, we will define

$$g_n(x, x^2) = (1/(r + h_2))\{r - c_1 - F(x)[(h_1 - h_2) - \alpha c_1] - \alpha \int_x^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + x^2 - t) f(t)\, dt\}.$$

Note that $g_1(x, x^2) = g(x)$ independent of x^2.

The preliminary results which are needed for the proof of Theorem 4.1 are listed in Appendix A. The proof of the theorem is presented in Appendix B, but for those sections of the proof that are similar to Theorem 2.2 of [5], only an outline of the methodology is included. A complete statement of the twenty-six results required for a rigorous induction argument will be given.

THEOREM 4.1. *Given assumptions* (1)–(7) *and* (i)–(iv), *the following hold for* $1 \leqq n \leqq N$:

(1) $B_n(\mathbf{x}, y, z)$ *is convex in* (y, z) *for all* $\mathbf{x}$.

(2) (a) *There exist unique continuously differentiable functions* $y_n(\mathbf{x})$, $z_n(\mathbf{x})$ *which minimize* $B_n(\mathbf{x}, y, z)$ *and solve*

$$B_n^{(m)}(\mathbf{x}, y_n(\mathbf{x}), z_n(\mathbf{x})) = 0,$$

$$B_n^{(m+1)}(\mathbf{x}, y_n(\mathbf{x}), z_n(\mathbf{x})) = 0;$$

(b) *There exists a unique continuously differentiable function* $p_n(\mathbf{x}, x^2)$ *which solves*

$$B_n^{(m)}(\mathbf{x}, p_n(\mathbf{x}, x^2), x^2) = 0.$$

(3) (a) *If* $x^2 < z_n(\mathbf{x})$, *then it is optimal to order* $y_n(\mathbf{x}) > 0$ *of product* 1 *and* $z_n(\mathbf{x}) - x^2$ *of product* 2; (Region I)

(b) *If* $x^2 \geqq z_n(\mathbf{x})$ *and* $F(x + x^2) < g_n(x, x^2)$, *it is optimal to order* $p_n(\mathbf{x}, x^2) > 0$ *of product* 1 *and none of product* 2. (Region II)

(c) *If* $F(x + x^2) \geqq g_n(x, x^2)$, *then it is optimal not to order.* (Region III)

(4) (a) *If* $(\mathbf{x}, x^2) \in$ *Region* I, *then* $F(x + y_n(\mathbf{x}) + z_n(\mathbf{x})) < g_n(x + y_n(\mathbf{x}), z_n(\mathbf{x}))$;

(b) *If* $(\mathbf{x}, x^2) \in$ *Region* II, *then* $F(x + p_n(\mathbf{x}, x^2) + x^2) < g_n(x + p_n(\mathbf{x}, x^2), x^2)$;

(c) $g_n(\mathbf{x}, x^2) \leqq g_{n-1}(\mathbf{x}, x^2)$ *for* $n \geqq 2$ *and all* $(\mathbf{x}, x^2)$.

(5) (a) $-1 \leqq y_n^{(1)}(\mathbf{x}) \leqq \cdots \leqq y_n^{(m-1)}(\mathbf{x}) < 0$ *in Region* I;

(b) $-1 \leqq z_n^{(m-1)}(\mathbf{x}) \leqq \cdots \leqq z_n^{(1)}(\mathbf{x}) < 0$ *in Region* I;

(c) $-1 \leqq p_n^{(1)}(\mathbf{x}, x^2) \leqq \cdots \leqq p_n^{(m)}(\mathbf{x}, x^2) < 0$ *in Region* II.

(6) (a)

$$C_n^{(i,j)}(\mathbf{x}, x^2) - C_n^{(i,m)}(\mathbf{x}, x^2) \geqq -\theta \sum_{k=1}^{i} G_{m-k}^{(j-k+1)}(\mathbf{x}(m-k)) \cdot \left[1 - \sum_{q=1}^{k-1} H_q(x_{m-k+q}, \cdots, x_{m-k+1})\right]$$

for $1 \leqq i \leqq j \leqq m - 1$, $(\mathbf{x}, x^2) \in$ *Regions* I *and* II *and* $(\mathbf{x}, x^2) \in$ *Region* III *such that* $(\bar{\mathbf{x}}(m - i), \mathbf{0}, x^2) \in$ *Regions* I *or* II. (For $i = 1$, the inequality is valid for all $(\mathbf{x}, x^2)$).

The following inequalities hold for $(\mathbf{x}, x^2) \in$ *Regions* I *or* II:

(b) $$C_n^{(i,j)}(\mathbf{x}, x^2) - C_n^{(i-1,j)}(\mathbf{x}, x^2) - C_n^{(i,m)}(\mathbf{x}, x^2) + C_n^{(i-1,m)}(\mathbf{x}, x^2)$$
$$\geqq -\theta G_{m-i}^{(j-i+1)}(\mathbf{x}(m-i)) \cdot \left[1 - \sum_{k=1}^{i-1} H_k(x_{m-i+k}, \cdots, x_{m-i-1})\right],$$
$$1 \leqq i \leqq j \leqq m-1,$$

(c) $$C_n^{(1,i)}(\mathbf{x}, x^2) - C_n^{(1,i-1)}(\mathbf{x}, x^2) \leqq \theta[G_{m-1}^{(i-1)}(\mathbf{x}) - G_{m-1}^{(i)}(\mathbf{x})],$$
$$2 \leqq i \leqq m-1;$$

(d) $$C_n^{(i,j)}(\mathbf{x}, x^2) - C_n^{(i-1,j)}(\mathbf{x}, x^2) - C_n^{(i,j-1)}(\mathbf{x}, x^2) + C_n^{(i-1,j-1)}(\mathbf{x}, x^2)$$
$$\leqq \theta[G_{m-i}^{(j-i)}(\mathbf{x}(m-i)) - G_{m-i}^{(j-i+1)}(\mathbf{x}(m-i))]$$
$$\cdot \left[1 - \sum_{k=1}^{i-1} H_k(x_{m-i+k}, \cdots, x_{m-i+1})\right], \qquad 2 \leqq i < j \leqq m-1;$$

(e) $$C_n^{(i,m)}(\mathbf{x}, x^2) \begin{cases} = 0 & \textit{for } (\mathbf{x}, x^2) \in \textit{Regions I or II}, \\ > 0 & \textit{for } (\mathbf{x}, x^2) \in \textit{Region III}, \end{cases}$$

and $C_n^{(i,m)}(\mathbf{x}, x^2) = C_n^{(m,m)}(\mathbf{x}, x^2)$ *for all* $(\mathbf{x}, x^2)$.

(7) (a) $$-c_1 - \theta \sum_{j=1}^{i} G_{m-j}(\mathbf{x}(m-j)) \cdot \left[1 - \sum_{k=1}^{j-1} H_k(x_{m-j+k}, \cdots, x_{m-j-1})\right]$$
$$\leqq C_n^{(i)}(\mathbf{x}, x^2) \leqq 0$$

for all $(\mathbf{x}, x^2) \in$ *Regions* I *and* II *for* $1 \leqq i \leqq m-1$. *In addition, the lower bound is valid for all* $(\mathbf{x}, x^2)$ *when* $i = 1$:

(b) $$C_n^{(i)}(\mathbf{x}, x^2) - C_n^{(j)}(\mathbf{x}, x^2)$$
$$\leqq c_1 + \theta \sum_{k=j+1}^{i} G_{m-k}(\mathbf{x}(m-k))$$
$$\cdot \left[\sum_{q=k-j}^{k-1} H_q(x_{m-k+q}, \cdots, x_{m-k+1})\right]$$

for $(\mathbf{x}, x^2) \in$ *Regions* I *and* II *only for* $1 \leqq j < i \leqq m-1$:

(c) $$C_n^{(i)}(\mathbf{x}, x^2) - C_n^{(j)}(\mathbf{x}, x^2) \geqq -\theta \sum_{k=j+1}^{i} G_{m-k}(\mathbf{x}(m-k))$$
$$\cdot \left[1 - \sum_{q=1}^{k-1} H_q(x_{m-k+q}, \cdots, x_{m-k-1})\right]$$

for $(\mathbf{x}, x^2) \in$ *Regions* I *or* II *and* $(\mathbf{x}, x^2) \in$ *Region* III *such that* $(\bar{\mathbf{x}}(m-i), \mathbf{0}, x^2) \in$ *Regions* I *or* II.

(8) (a) $$\lim_{x_{m-i} \to 0} \{C_n^{(i)}(\mathbf{x}, x^2) - C_n^{(i-1)}(\mathbf{x}, x^2)\} = 0, \qquad 2 < i \leqq m-1, \textit{ all } (\mathbf{x}, x^2);$$

(b) $$\lim_{x_{m-1} \to 0} \{C_n^{(m)}(\mathbf{x}, x^2) - C_n^{(1)}(\mathbf{x}, x^2)\} \geqq c_1 - c_2 \qquad \textit{for all } (\mathbf{x}, x^2)$$

with equality holding if and only if $(\mathbf{x}, x^2) \in$ *Region* I:

(c) $$\lim_{x_{m-1}\to 0} C_n^{(1)}(\mathbf{x}, x^2) \begin{cases} = -c_1 & \text{if } (\mathbf{x}, x^2) \in \text{Regions I or II}, \\ > -c_1 & \text{if } (\mathbf{x}, x^2) \in \text{Region III}; \end{cases}$$

(d) $$C_n^{(m)}(\mathbf{x}, x^2) \begin{cases} = -c_2 & \text{if } (\mathbf{x}, x^2) \in \text{Region I}, \\ > -c_2 & \text{if } (\mathbf{x}, x^2) \in \text{Regions II or III}; \end{cases}$$

(e) $$C_n^{(i)}(\mathbf{0}, z) - C_n^{(m)}(\mathbf{0}, z) \geqq 0, \qquad 1 \leqq i \leqq m-1 \quad \text{and all } z.$$

(f) $$C_n^{(m)}(\mathbf{x}, x^2) \geqq C_{n-1}^{(m)}(\mathbf{x}, x^2) \quad \text{for all } (\mathbf{x}, x^2) \quad \text{and } n \geqq 1.$$

The theorem reveals a number of structural properties of the optimal policy. From (5) we see that all three ordering functions decrease in each variable. Both (5)(a) and (5)(c) show that the optimal quantity of perishable inventory ordered will decrease more significantly if the stocks of newer inventory are increased than if stocks of older inventory are increased. Since z_n represents the quantity of nonperishable stock after ordering, it is not surprising that the ordering of the partial derivatives should go in the opposite direction. That the total system policy exhibits a critical number property is shown in the following Corollary.

COROLLARY 4.2. *If* $(\mathbf{x}, x^2) \in$ *Region* I *and* n *periods remain, then* $x + y_n(\mathbf{x}) + z_n(\mathbf{x}) = u_n^*$ *independent of* $(\mathbf{x}, x^2)$. *Furthermore,* $u_{n+1}^* \leqq u_n^*$, $n = 1, 2, \cdots, N$.

Proof. The $n = 1$ case has been established in the previous section, where $u_1^* = u^*$.

From the proof of (5)(b) of Theorem 4.1, $y_n^{(i)}(\mathbf{x}) + z_n^{(i)}(\mathbf{x}) = -1$ in Region I, from which it follows that $x + y_n(\mathbf{x}) + z_n(\mathbf{x}) = u_n^*$. The fact that the numbers u_n^* are nonincreasing in n follows directly from the second defining equation for y_n and z_n in section (2)(a) and the inequality of section (8)(f). □

The relationship between the functions y_n and p_n is now shown.

COROLLARY 4.3. *Suppose that* $(\mathbf{x}, x^2) \in$ *Region* II *and* n *periods remain. Then* $p_n(\mathbf{x}, x^2) \leqq y_n(\mathbf{x})$. □

Proof. By construction, $p_n(\mathbf{x}, z_n(\mathbf{x})) = y_n(\mathbf{x})$. If $(\mathbf{x}, x^2) \in$ Region II, then $x^2 \geqq z_n(\mathbf{x})$ and the result follows from (5)(c). □

It is somewhat surprising that the boundaries separating the ordering regions should change with time. In both the multiproduct nonperishable inventory problem and the single product perishable inventory problem, when costs and demands are stationary, the salvage value assumption guarantees that the ordering regions are stationary as well. The nonstationarity in our case results from the fact that when $(\mathbf{x}, x^2) \in$ Region I, one orders not to the boundary of Regions I and II, but into the interior of Region II (see Fig. 1). The salvage value assumption is still necessary to insure that the function defining the boundary between Regions II and III depends on the vector $\mathbf{x}$ only through the sum x.

The result of (4) in some circumstances provides an upper bound on the total quantity of perishable inventory in stock. Suppose $(\mathbf{x}, x^2) \in$ Region II when n periods remain and $g(x) < 1$. Then from (4)(b), (4)(c) and the fact that $g_n(x, x^2)$ is nonincreasing in both arguments, it follows that $x + p_n(\mathbf{x}, x^2) < F^{-1}(g_n(x, x^2)) - x_2 \leqq F^{-1}(g(x)) - x^2$.

Another significant result can be inferred from (4)(a), (b) and (c). If $(\mathbf{x}, x^2) \in$ Regions I or II in any period, then it must be true that $(\mathbf{x}, x^2) \in$ Regions I or II for all subsequent periods. That is, if it is optimal to order in product 1 in any period, it is optimal to order in product 1 for each subsequent period.

5. Generalizations. For the sake of convenience, we have assumed that excess demand is backlogged in product 2. However, since $c_1 < c_2$ it is far more realistic to assume that excess demand is backlogged in the first product. In that case, the first component of $\mathbf{s}_1(\mathbf{x}, y, z, t)$ becomes

$$s_{1,m-1}(\mathbf{x}, y, z, t) = \begin{cases} (y - (t - x)^+)^+ & \text{if } x < t \leqq x + y, \\ 0 & \text{if } x + y < t < x + y + z, \\ x + y + z - t & \text{if } x + y + z < t, \end{cases}$$

and the remaining components are unchanged. Also

$$s_2(x + y, z, t) = (z - (t - x - y)^+)^+.$$

In this case we obtain

$$\begin{aligned} B_n^{(m)}(\mathbf{x}, y, z) = L^{(m)}(\mathbf{x}, y, z) &+ \alpha \int_0^{x+y} C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y, z, t), z) f(t)\, dt \\ &+ \alpha \int_{x+y}^{x+y+z} C_{n-1}^{(m)}(\mathbf{0}, x + y + z - t) f(t)\, dt \\ &+ \alpha \int_{x+y+z}^{\infty} C_{n-1}^{(1)}(x + y + z - t, \mathbf{0}, 0) f(t)\, dt \end{aligned}$$

and

$$\begin{aligned} B_n^{(m+1)}(\mathbf{x}, y, z) = L^{(m+1)}(\mathbf{x}, y, z) & \\ &+ \alpha \int_0^{x+y+z} C_{n-1}^{(m)}(\mathbf{s}_1(\mathbf{x}, y, z, t), s_2(x + y, z, t)) f(t)\, dt \\ &+ \alpha \int_{x+y+z}^{\infty} C_{n-1}^{(1)}(x + y + z - t, \mathbf{0}, 0) f(t)\, dt, \end{aligned}$$

from which it follows that

$$u^* = F^{-1}\left[\frac{r - c_2 + \alpha c_1}{r + h_2 + \alpha(c_1 - c_2)}\right].$$

The constant w^* does not change. However, the function defining the boundary between Regions II and III becomes

$$\begin{aligned} & g_n(x, x^2) \\ &= (1/(r + h_2))\Big\{r - c_1(1 - \alpha) - F(x)[(h_1 - h_2)] - \alpha c_1(F(x + x^2) - F(x)) \\ &\qquad - \alpha \int_x^{x+x^2} C_{n-1}^{(m)}(\mathbf{0}, x + x^2 - t) f(t)\, dt\Big\}. \end{aligned}$$

All of the results of Theorem 4.1 will now follow with the changes above.

We may also assume that demand is not backlogged, but lost. The only change required for lost sales from backlogging in product 1 is that

$$s_{1,m-1}(\mathbf{x}, y, t) = (y - (t - x)^+)^+,$$

which results in obvious changes in the various constants and region boundaries.

Another assumption which can be relaxed is that all costs are linear. The analysis remains essentially identical if we assume that the holding and shortage costs are convex nondecreasing functions vanishing at the origin.

Our methods of analysis may be applied to the case of nonstationary demands only when the demand distribution is stochastically increasing over time. Although Theorem A.1 remains valid for general nonstationary demands [5], Theorem 4.1 will not. The problem that arises is that (4)(a), (b) and (c) will not necessarily imply that $(\mathbf{s}_1(\mathbf{x}, y, t), s_2(x + y, z, t))$ is in Regions I or II when $n - 1$ periods remain if $(\mathbf{x}, x^2) \in$ Regions I or II when n periods remain unless demands increase stochastically. A similar phenomenon occurs in the single product case.

Appendix A. Preliminary results.

THEOREM A.1. *Define* $s_i(\mathbf{x}(n), t) = (x_{i+1} - (t - \sum_{j=1}^{i} x_j)^+)^+$ *for* $1 \leqq i \leqq m - 1$, *where* $\mathbf{x}(n) = (x_n, \cdots, x_1)$. *Letting* $\mathbf{s}(\mathbf{x}(n), t) = (s_{n-1}(\mathbf{x}(n), t), \cdots, s_1(\mathbf{x}(n), t)$, *we have that*

$$G_n(\mathbf{x}(n)) = \int_0^{w_n} G_{n-1}(\mathbf{s}(\mathbf{x}(n), t)) f(t)\, dt, \qquad 1 \leqq n \leqq m,$$

where $w_n = \sum_{j=1}^{n} x_j$ *and* $G_0(\cdot) = 1$.

The analysis requires an exhaustive enumeration of various properties of the G_n functions and their derivatives. It becomes convenient to introduce another class of functions H_n which exhibit a type of duality relationship with G_n.

Define for $2 \leqq j \leqq n$, $2 \leqq n \leqq m$,

$$H_j(\bar{\mathbf{x}}(n - j + 1)) = \int_0^{x_n} H_{j-1}(v + x_{n-1}; x_{n-2}, \cdots, x_{n-j+1}) f(x_n - v)\, dv$$
$$- F(x_n) H_{j-1}(x_{n-1}, \cdots, x_{n-j+1}),$$

where $\bar{\mathbf{x}}(n - j + 1) = (x_n, \cdots, x_{n-j+1})$ (a vector of dimension j formed from the first j coordinates of $\mathbf{x}(n)$). The following lemmas are required in the proof of Theorem 4.1.

LEMMA A.1.

$$\frac{\partial}{\partial x_{m-i}}\left[\int_0^{x_m} G_m(u; \mathbf{x})\, du\right] = G_m(\mathbf{x}) - \sum_{j=1}^{i} G_{m-j}(\mathbf{x}(m - j)) H_j(\bar{\mathbf{x}}(m - j))$$

for $1 \leqq i \leqq m - 1$.

LEMMA A.2.

$$H_j^{(1)}(\mathbf{x}(j)) = G_j^{(j)}(\mathbf{x}(j)) \quad \textit{for } 1 \leqq j \leqq m.$$

LEMMA A.3.

$$H_j(\bar{\mathbf{x}}(m-j+1)) \leqq 1 - \sum_{k=1}^{j-1} H_k(x_{m-j+k}, \cdots, x_{m-j+1}) \quad \textit{for } 1 \leqq j \leqq m.$$

LEMMA A.4. (i) $G_k(\mathbf{x}(k)) \leqq G_{k-1}(x_k, \cdots, x_2)$,
(ii) $H_k(\mathbf{x}(k)) \leqq H_{k-1}(x_k, \cdots, x_2)$,
(iii) $G_k(\mathbf{x}(k)) \geqq H_k(\mathbf{x}(k))$ *for* $1 \leqq k \leqq m$.

Appendix B. Proof of Theorem 4.1. Many of the results have already been established for the $n = 1$ case. The remainder follow in a similar fashion to the general argument.

Assume all results hold for periods $1, 2, \cdots, n-1 < N$. We prove them true for n.

$$\begin{aligned}
B_n^{(m,m)}(\mathbf{x}, y, z) &= (h_1 - h_2)f(x+y) + \theta G_m^{(1)}(y;\mathbf{x}) + (r+h_2)f(x+y+z) \\
&\quad + \alpha \int_0^{x+y} C_{n-1}^{(1,1)}(\mathbf{s}_1(\mathbf{x}, y, t), z) f(t)\,dt \\
&\quad + \alpha f(x+y)[C_{n-1}^{(1)}(\mathbf{0}, z) - C_{n-1}^{(m)}(\mathbf{0}, z)] \\
&\quad + \alpha \int_{x+y}^{\infty} C_{n-1}^{(m,m)}(\mathbf{0}, x+y+z-t) f(t)\,dt \qquad (1)\\
&\geqq (h_1 - h_2)f(x+y) + \theta G_m^{(1)}(y;\mathbf{x}) + (r+h_2)f(x+y+z) \\
&\quad - \alpha\theta \int_0^{x+y} G_{m-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y, t)) f(t)\,dt
\end{aligned}$$

by the inductive assumption on (6)(a), (6)(e) and (8)(e),

$$= (h_1 - h_2)f(x+y) + \theta(1-\alpha)G_m^{(1)}(y;\mathbf{x}) + (r+h_2)f(x+y+z) \geqq 0$$

by Theorem A.1 and assumption (i) of § 3.

$$\begin{aligned}
B_n^{(m+1,m+1)}(\mathbf{x}, y, z) &= (r+h_2)f(x+y+z) + \alpha \int_0^{x+y} C_{n-1}^{(m,m)}(\mathbf{s}_1(\mathbf{x}, y, t), z) f(t)\,dt \\
&\quad + \alpha \int_{x+y}^{\infty} C_{n-1}^{(m,m)}(\mathbf{0}, x+y+z-t) f(t)\,dt \geqq 0
\end{aligned}$$

by (6)(e).

Also since $B_n^{(m,m+1)}(\mathbf{x}, y, z) = B_n^{(m+1,m+1)}(\mathbf{x}, y, z)$ everywhere (by the inductive assumption on (6)(e)), a necessary and sufficient condition for convexity is that $B_n^{(m,m)}(\mathbf{x}, y, z) - B_n^{(m,m+1)}(\mathbf{x}, y, z) \geqq 0$. Forming the difference, we obtain

$$\begin{aligned}
&(h_1 - h_2)f(x+y) + \theta G_m^{(1)}(y;\mathbf{x}) + \alpha \int_0^{x+y} \{C_{n-1}^{(1,1)}(s_1(\mathbf{x}, y, t), z) \\
&\quad - C_{n-1}^{(1,m)}(s_1(\mathbf{x}, y, t), z)\} f(t)\,dt + \alpha f(x+y)[C_{n-1}^{(1)}(\mathbf{0}, z) - C_{n-1}^{(m)}(\mathbf{0}, z)] \geqq 0
\end{aligned}$$

from the inductive assumption on (6)(a) and (8)(e) and Theorem A.1 (as above).

(2) (a). The minimizing point $(y_n(\mathbf{x}), z_n(\mathbf{x}))$ will now satisfy the equations

$$B_n^{(m)}(\mathbf{x}, y_n(\mathbf{x}), z_n(\mathbf{x})) = B_n^{(m+1)}(\mathbf{x}, y_n(\mathbf{x}), z_n(\mathbf{x})) = 0.$$

For the sake of convenience, we will write y_n and z_n. It is understood that these are both functions on R^{m-1}. The equations defining y_n and z_n are

$$c_1 + (h_1 - h_2)F(x + y_n) + \theta G_m(y_n; \mathbf{x}) - r + (r + h_2)F(x + y_n + z_n)$$
$$+ \alpha \int_0^{x+y_n} C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) f(t)\, dt + \alpha \int_{x+y_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + y_n + z_n - t) f(t)\, dt = 0$$

and

$$c_2 - r + (r + h_2)F(x + y_n + z_n) + \alpha \int_0^{\infty} C_{n-1}^{(m)}(\mathbf{s}_1(\mathbf{x}, y_n, t), s_2(x + y_n, z, t)) f(t)\, dt = 0.$$

The principal minors of the Hessian will be strictly positive in a neighborhood of (y_n, z_n) if $y_n > 0$ since the definition of G_m and assumption (3) guarantee that $y_n > 0$ implies $G_m^{(1)}(y_n; \mathbf{x}) > 0$. Now $y_n > 0$ if and only if $B_n^{(m)}(\mathbf{x}, 0, z_n) < 0$. Solving for $F(x + y_n + z_n)$ in the second equation and substituting into the first, we obtain

$$\begin{aligned} B_n^{(m)}(\mathbf{x}, 0, z_n(\mathbf{x})) &= c_1 + (h_1 - h_2)F(x) - c_2 + \alpha \int_0^{x} (C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, 0, t), z_n) \\ &\quad - C_{n-1}^{(m)}(\mathbf{s}_1(\mathbf{x}, 0, t) z_n)) f(t)\, dt \\ &\leqq c_1(1 - \alpha F(x)) - c_2(1 - \alpha F(x)) + (h_1 - h_2)F(x) < 0 \end{aligned}$$

by the inductive assumption on (8)(b). The negativity follows from assumptions (ii) and (iv) pf § 3.

(b) As above, if $p_n(\mathbf{x}, x^2)$ satisfies $B_n(\mathbf{x}, p_n(\mathbf{x}, x^2), x^2) = \min_y \{B_n(\mathbf{x}, y, x^2)\}$, then it is the solution to the equation $B_n^{(m)}(\mathbf{x}, p_n(\mathbf{x}, x^2), x^2) = 0$. Again, for convenience, we will write p_n, which is understood to be a function on R^m. Uniqueness will follow if $B_n^{(m,m)}(\mathbf{x}, p_n, x^2) > 0$, which will hold if either of the conditions $x + p_n + x_2 > 0$ or $p_n > 0$. Suppose that $p_n \leqq 0$. From the definition of p_n we obtain

$$\begin{aligned} (r + h_2)F(x + p_n + x^2) &= -c_1 + (h_2 - h_1)F(x + p_n) - \theta G_m(p_n; \mathbf{x}) + r \\ &\quad - \alpha \int_0^{x+p_n} C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, p_n, t), x^2) f(t)\, dt \\ &\quad - \alpha \int_{x+p_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + p_n + x^2 - t) f(t)\, dt \\ &\geqq -c_1(1 - \alpha F(x + p_n)) + (h_2 - h_1)F(x + p_n) \\ &\quad + r + \alpha c_2(1 - F(x + p_n)) - \theta(1 - \alpha) G_m(p_n; \mathbf{x}) \end{aligned}$$

by (7)(a) and (8)(d),

$$\begin{aligned} &> (c_2 - c_1)(1 - \alpha F(x + p_n)) + (h_2 - h_1)F(x + p_n) \\ &> 0 \end{aligned}$$

by assumption (iii) and the fact that $p_n \leqq 0$ implies $G_m(p_n; \mathbf{x}) = 0$. The positivity follows from assumptions (i) and (iii).

The fact that the functions y_n, z_n and p_n are continuously differentiable follows from the implicit function theorem because of strict convexity in a neighborhood of these points (see [5], for example).

(3) (a) In this case, the global minimum of $B_n(\mathbf{x}, y, z)$ can be attained so it is clearly optimal to do so.

(b) In this case, it is not possible to order to the global minimum. However, if $B_n^{(m)}(\mathbf{x}, 0, x^2) < 0$, then it is optimal to order p_n (where $p_n > 0$). Since

$$B_n^{(m)}(\mathbf{x}, 0, x^2) = (r + h_2)(F(x + x^2) - g_n(x, x^2)),$$

the characterization follows.

(c) From (b), $p_n \leqq 0$ if and only if $F(x + x^2) \geqq g_n(\mathbf{x}, x^2)$.

(4) The proofs of cases (a) and (b) are essentially identical with the notation adjusted appropriately. We prove (b). (For (a) one substitutes y_n for p_n and z_n for x^2.) If $(\mathbf{x}, x^2) \in$ Region II, then

$$\begin{aligned} 0 = B_n^{(m)}(\mathbf{x}, p_n, x^2) &= c_1 + (h_1 - h_2)F(x + p_n) + \theta G_m(p_n; \mathbf{x}) - r \\ &\quad + (r + h_2)F(x + x^2 + p_n) \\ &\quad + \alpha \int_0^{x + p_n} C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, p_n, t), x^2) f(t)\, dt \\ &\quad + \alpha \int_{x + p_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + x^2 + p_n - t) f(t)\, dt \\ &\geqq c_1 + (h_1 - h_2)F(x + p_n) + \theta(1 - \alpha)G_m(p_n, \mathbf{x}) - r \\ &\quad + (r + h_2)F(x + x^2 + p_n) - \alpha c_1 F(x + p_n) \\ &\quad + \alpha \int_{x + p_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + x^2 + p_n - t) f(t)\, dt \end{aligned}$$

from the inductive assumption on (7)(a) and Theorem A.1.

Dropping the term $\theta(1 - \alpha)G_m(p_n; \mathbf{x})$, which is positive, we obtain

$$F(x + x^2 + p_n) < g_n(x + p_n, x^2),$$

as required. (4)(c) follows directly from the inductive assumption on (8)(f).

Taken together, (4)(a), (b) and (c) imply that if $(\mathbf{x}, x^2) \in$ Regions I or II when n periods remain, then the inventory on hand for the next period, $(\mathbf{s}_1(\mathbf{x}, y_n, t), s_2(x + y_n, z_n, t))$ (or $(\mathbf{s}_1(\mathbf{x}, p_n, t), s_2(x + p_n, x^2, t))$) will also lie in Regions I or II. This follows since, for the second case, the sum of the components of $(\mathbf{s}_1(\mathbf{x}, p_n, t), s_2(x + p_n, x^2, t))$ is $x + x^2 + p_n - t$, for some $t \geqq 0$. It is easily verified that $g_n(x, x^2)$ is nonincreasing in both of its arguments so that

$$F(x + x^2 + p_n - t) < g_n(x + p_n - t, x^2) \leqq g_{n-1}(x + p_n - t, x^2).$$

A similar argument applies when $(\mathbf{x}, x^2) \in$ Region I.

(5) (a) Suppose $(\mathbf{x}, x^2) \in$ Region I. Combining the equations defining y_n and z_n, we obtain

$$c_1 + (h_1 - h_2)F(x + y_n) - c_2 + \theta G_m(y_n; \mathbf{x}) + \alpha \int_0^{x+y_n} \{C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) - C_{n-1}^{(m)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n)\} f(t)\, dt = 0.$$

Differentiating implicitly with respect to x_{m-i}, we have

$$y_n^{(i)}(\mathbf{x}) = -\frac{N_i(y_n(\mathbf{x}); \mathbf{x})}{D(y_n(\mathbf{x}); \mathbf{x})},$$

where

$$\begin{aligned} N_i(y_n; \mathbf{x}) = {} & (h_1 - h_2) f(x + y_n) + \theta G_m^{(i+1)}(y_n; \mathbf{x}) \\ & + \alpha \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{(1,i+1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) \\ & - C_{n-1}^{(m,i+1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n)\} f(t)\, dt \\ & + \alpha f(x + y_n)(C_{n-1}^{(1)}(\mathbf{0}, z_n) - C_{n-1}^{(m)}(\mathbf{0}, z_n)) \end{aligned}$$

and

$$\begin{aligned} D(y_n; \mathbf{x}) = {} & (h_1 - h_2) f(x + y_n) + \theta G_m^{(1)}(y_n; \mathbf{x}) \\ & + \sum_{j=1}^{m} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{(1,1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) \\ & - C_{n-1}^{(m,1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n)\} f(t)\, dt \\ & + \alpha f(x + y_n)(C_{n-1}^{(1)}(\mathbf{0}, z_n) - C_{n-1}^{(m)}(\mathbf{0}, z_n)). \end{aligned}$$

Note that the resulting term involving $z_n^{(i)}$ drops out by the inductive assumption on (6)(e). By the result of section (4) above, $(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) \in$ Regions I or II for all $t \geqq 0$, so that we may apply the inductive assumptions on (6)(b) and (6)(a) respectively to obtain

$$N_i(y_n; \mathbf{x}) \geqq (h_1 - h_2) f(x + y_n) + \theta(1 - \alpha) G_m^{(i+1)}(y_n; \mathbf{x}) > 0, \qquad 1 \leqq i \leqq m - 2,$$

$$D(y_n; \mathbf{x}) \geqq (h_1 - h_2) f(x + y_n) + \theta(1 - \alpha) G_m^{(1)}(y_n; \mathbf{x}) > 0.$$

One then shows $D(y_n; \mathbf{x}) = N_0(y_n; \mathbf{x}) \geqq N_1(y_n; \mathbf{x}) \geqq \cdots \geqq N_{m-1}(y_n; \mathbf{x})$ by applying the inductive assumption on (6)(b). The logic is essentially identical to the single product case and can be found in [5].

(b) From the equation

$$c_2 - r + (r + h_2)F(x + y_n + z_n) + \alpha \int_0^{x+y_n} C_{n-1}^{(m)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) f(t)\, dt + \alpha \int_{x+y_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + y_n + z_n - t) f(t)\, dt = 0,$$

differentiating implicitly with respect to x_{m-i} and applying the inductive assumption on (6)(c), it follows that

$$y_n^{(i)} + z_n^{(i)} = -1$$

and (5)(b) follows from (5)(a).

(c) The logic here is identical to (5)(a) for $1 \leqq i \leqq m-1$ and a similar argument holds for the case $i = m$.

(6), (7), (8). The proofs of the results of sections (6), (7) and (8) are somewhat tedious and involve the application of the appropriate inductive assumptions and Theorem A.1 at each stage. The logic is essentially the same as it is for the single product model ([4] and [5]). Suppose that $(\mathbf{x}, x^2) \in$ Region I. Then

$$C_n(\mathbf{x}, x^2) = B_n(\mathbf{x}, y_n, z_n) - c_2 x^2.$$

Differentiating with respect to x_{m-i} for $1 \leqq i \leqq m-1$, we obtain

$$\begin{aligned}
C_n^{(i)}(\mathbf{x}, x^2) &= B_n^{(i)}(\mathbf{x}, y_n, z_n) + B_n^{(m)}(\mathbf{x}, y_n, z_n) \cdot y_n^{(i)} + B_n^{(m+1)}(\mathbf{x}, y_n, z_n) \cdot z_n^{(i)} \\
&= B_n^{(i)}(\mathbf{x}, y_n, z_n) \quad \text{(by the definitions of } y_n \text{ and } z_n) \\
&= (h_1 - h_2)F(x + y_n) + \frac{\partial}{\partial x_{m-i}}\left[\int_0^{y_n} G_m(u:\mathbf{x})\,du\right] - r \\
&\quad + (r + h_2)F(x + y_n + z_n) + \alpha \int_0^{w_{m-i}} C_{n-1}^{(i+1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) f(t)\,dt \\
&\quad + \alpha \sum_{j=m-i+1}^{m} \int_{w_{j-1}}^{w_j} C_{n-1}^{(m-j+1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) f(t)\,dt \\
&\quad + \alpha \int_{x+y_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + y_n + z_n - t) f(t)\,dt \\
&= \Big\{(h_1 - h_2)F(x + y_n) + \theta G_m(y_n; \mathbf{x}) - r + (r + h_2)F(x + y_n + z_n) \\
&\qquad + \alpha \int_0^{x+y_n} C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) f(t)\,dt \\
&\qquad + \alpha \int_{x+y_n}^{\infty} C_{n-1}^{(m)}(\mathbf{0}, x + y_n + z_n - t) f(t)\,dt\Big\} \\
&\quad - \theta \sum_{j=1}^{i} G_{m-j}(\mathbf{x}(m-j)) H_j(y_n; \bar{\mathbf{x}}(m-j)) \\
&\quad + \alpha \int_0^{w_{m-i}} \{C_{n-1}^{(i+1)}(\mathbf{s}_1(\mathbf{x}, y_n, t) z_n) - C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n)\} f(t)\,dt \\
&\quad + \alpha \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{(m-j+1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n) \\
&\qquad\qquad - C_{n-1}^{(1)}(\mathbf{s}_1(\mathbf{x}, y_n, t), z_n)\} f(t)\,dt
\end{aligned}$$

which results from Lemma A.1. The expression in brackets is exactly $B_n^{(m)}(\mathbf{x}, y_n, z_n) = 0$. In the case where $(\mathbf{x}, x^2) \in$ Region II, one obtains an identical expression except that p_n is substituted for y_n and x^2 for z_n. Using the remaining terms, the various sums and differences of the first and second cross partials of $C_n(\mathbf{x}, x^2)$ are formed. Each of the inequalities is established by using the appropriate inductive assumptions as given by the following chart.

To prove	*Use the inductive assumption on*
(6)(a)	(6)(b) and (6)(c) if $i < m - 1$ and (6)(b) and (7)(a) if $i = m - 1$ for $(\mathbf{x}, x^2) \in$ Regions I or II; (6)(a) if $i < m - 1$ and (6)(a) and (7)(a) if $i = j = m - 1$ for $(\mathbf{x}, x^2) \in$ Region III.
(6)(b)	(6)(b) and (6)(c) if $i < m - 1$. (6)(a) and (7)(a) if $i = j = m - 1$.
(6)(c)	(6)(c) and (6)(d) if $i < m - 1$. (6)(b) if $i = m - 1$.
(6)(d)	(6)(c) and (6)(d).
(7)(a)	(7)(c) if $i < m - 1$. (7)(a) and (7)(c) if $i = m - 1$.
(7)(b)	(7)(b) if $i < m - 1$. (7)(a) and (7)(b) if $i = m - 1$.
(7)(c)	(7)(c).
(8)(a)	(8)(a).
(8)(b)	(8)(a) and (8)(b).
(8)(c)	(8)(a) and (8)(c).
(8)(d)	(8)(d).
(8)(e)	(8)(e).
(8)(f)	(8)(f).

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